DAY NINTEEN

Unit Test 2

(Calculus)

1 Let $f:(2,3)\to(0,1)$ be defined by $f(x)=x-[x]$, then $f^{-1}(x)$ is equal to

(a) $x-2$ (b) $x+1$ (c) $x-1$ (d) $x+2$

2 $\int \frac{dx}{\sin x-\cos x+\sqrt{2}}$ is equal to

(a) $-\frac{1}{\sqrt{2}}\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$ (b) $\frac{1}{\sqrt{2}}\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$

(c) $\frac{1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$ (d) $-\frac{1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$

3 $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}$ is equal to

(a) e (b) e^2 (c) e^{-1} (d) 1

4 If the normal to the curve $y=f(x)$ at the point $(3,4)$ makes an angle $3\pi/4$ with the positive X-axis, then $f'(3)$ is equal to

(a) -1 (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) 1

5 The area bounded by $y=\sin^{-1}x$, $x=1/\sqrt{2}$ and X-axis is

(a) $\left(\frac{1}{\sqrt{2}}+1\right)$ sq units (b) $\left(1-\frac{1}{\sqrt{2}}\right)$ sq units

(c) $\frac{\pi}{4\sqrt{2}}$ sq units (d) $\left(\frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right)$ sq units

6 The value of $\int_{-\pi/2}^{\pi/2}\log\left(\frac{2-\sin\theta}{2+\sin\theta}\right)d\theta$ is

(a) 0 (b) 1 (c) 2 (d) None of these

7 The general solution of the differential equation $(2x-y+1)\,dx+(2y-x+1)\,dy=0$ is

(a) $x^2+y^2+xy-x+y=C$
(b) $x^2+y^2-xy+x+y=C$
(c) $x^2-y^2+2xy-x+y=C$
(d) $x^2-y^2-2xy+x-y=C$

8 The function $f(x)=x^4-\frac{x^3}{3}$ is

(a) increasing for $x>\frac{1}{4}$ and decreasing for $x<\frac{1}{4}$
(b) increasing for every value of x
(c) decreasing for every value of x
(d) None of the above

9 If $y=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}$, then $\frac{dy}{dx}$ is equal to

(a) y (b) $y+\frac{x^n}{n!}$ (c) $y-\frac{x^n}{n!}$ (d) $y-1-\frac{x^n}{n!}$

10 The value of $\lim_{x\to1}\frac{x^{1/4}-x^{1/5}}{x^3-1}$ is

(a) $\frac{1}{20}$ (b) $\frac{1}{40}$ (c) $\frac{1}{60}$ (d) $\frac{3}{20}$

11 The differential coefficient of the function $|x-1|+|x-3|$ at the point $x=2$ is

(a) -2 (b) 0
(c) 2 (d) not defined

12 The difference between the greatest and least values of the function $f(x)=\cos x+\frac{1}{2}\cos 2x-\frac{1}{3}\cos 3x$ is

(a) $\frac{2}{3}$ (b) $\frac{8}{7}$ (c) $\frac{9}{4}$ (d) $\frac{3}{8}$

13 In the mean value theorem $\frac{f(b)-f(a)}{b-a}=f'(c)$, if $a=0$, $b=\frac{1}{2}$ and $f(x)=x(x-1)(x-2)$, then value of c is

(a) $1-\frac{\sqrt{15}}{6}$ (b) $1+\sqrt{15}$ (c) $1-\frac{\sqrt{21}}{6}$ (d) $1+\sqrt{21}$

14 If the sides and angles of a plane triangle vary in such a way that its circumradius remains constant. Then, $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}$ is equal to (where, da, db and dc are small increments in the sides a, b and c, respectively).

(a) 1 (b) 2 (c) 0 (d) 3

15 $\int_0^{1.5}[x^2]\,dx$, where [.] denotes the greatest integer function, is equal to

(a) $2+\sqrt{2}$ (b) $2-\sqrt{2}$ (c) $-2+\sqrt{2}$ (d) $-2-\sqrt{2}$

16 The integrating factor of the differential equation $\frac{dy}{dx} = y\tan x - y^2 \sec x$, is

(a) $\tan x$ (b) $\sec x$ (c) $-\sec x$ (d) $\cot x$

17 The area bounded by the straight lines $x = 0$, $x = 2$ and the curve $y = 2^x$, $y = 2x - x^2$ is

(a) $\frac{4}{3} - \frac{1}{\log 2}$ (b) $\frac{3}{\log 2} + \frac{4}{3}$

(c) $\frac{4}{\log 2} - 1$ (d) $\frac{3}{\log 2} - \frac{4}{3}$

18 The area of the region bounded by $y = |x - 1|$ and $y = 1$ is

(a) 2 sq units (b) 1 sq unit

(c) 1/2 sq unit (d) None of these

19 If $I_1 = \int_{1-k}^{k} x \sin\{x(1-x)\}\,dx$ and

$I_2 = \int_{1-k}^{k} \sin\{x(1-x)\}\,dx$, then

(a) $I_1 = 2I_2$ (b) $2I_1 = I_2$

(c) $I_1 = I_2$ (d) None of these

20 If $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x f\{x(3-x)\}\,dx$

and $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f\{x(3-x)\}\,dx$, where f is a continuous function and z is any real number, then $\frac{I_1}{I_2}$ is equal to

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$

(c) 1 (d) None of these

21 If f and g are continuous functions on $[0, \pi]$ satisfying $f(x) + f(\pi - x) = g(x) + g(\pi - x) = 1$, then $\int_0^{\pi}[f(x) + g(x)]\,dx$ is equal to

(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{2}$

22 The function $f(x) = \begin{cases} |2x - 3| \cdot [x], & x \ge 1 \\ \sin\left(\frac{\pi x}{2}\right), & x < 1 \end{cases}$

(where, $[x]$ denotes the greatest integer $\le x$) is

(a) continuous at $x = 2$

(b) differentiable at $x = 1$

(c) continuous but not differentiable at $x = 1$

(d) None of the above

23 If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \dots \infty}}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{f'(x)}{2y-1}$ (b) $\frac{f'(x)}{2y+1}$ (c) $\frac{f'(x)}{1-2y}$ (d) $\frac{f'(x)}{4+2y}$

24 If $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real numbers x, then

(a) h is increasing whenever f is increasing

(b) h is increasing whenever f is decreasing

(c) h is decreasing whenever f is increasing

(d) Nothing can be said in general

25 If $u = \int e^{ax}\cos bx\,dx$ and $v = \int e^{ax}\sin bx\,dx$, then $(a^2 + b^2)(u^2 + v^2)$ is equal to

(a) $2e^{ax}$ (b) $(a^2 + b^2)e^{2ax}$

(c) e^{2ax} (d) $(a^2 - b^2)e^{2ax}$

26 If $\frac{d[f(x)]}{dx} = g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\cdot g(x)\,dx$ is equal to

(a) $f(b) - f(a)$ (b) $g(b) - g(a)$

(c) $\frac{[f(b)]^2 - [f(a)]^2}{2}$ (d) $\frac{[g(b)]^2 - [g(a)]^2}{2}$

27 If y be the function which passes through $(1, 2)$ having slope $(2x + 1)$. Then, the area bounded between the curve and X-axis is

(a) 6 sq units (b) $\frac{5}{6}$ sq unit

(c) $\frac{1}{6}$ sq unit (d) None of these

28 If $h(x) = \min\{x, x^2\}$, for every real number of x. Then,

(a) h is continuous for all x (b) h is differentiable for all x

(c) $h'(x) = 2$ for all $x > 1$ (d) None of these

29 The function $f(x) = \begin{cases} e^{2x} - 1, & x < 0 \\ ax + \frac{bx^2}{2} - 1, & x \ge 0 \end{cases}$ is continuous and differentiable for

(a) $a = 1, b = 2$ (b) $a = 2, b = 4$

(c) $a = 2$, any b (d) any a, $b = 4$

30 The value of $\lim_{x \to \frac{\pi}{2}} \frac{\int_{\frac{\pi}{2}}^{x} t\,dt}{\sin(2x - \pi)}$ is

(a) ∞ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

31 Let $f(x) = \left[\sqrt{2}\cos\left(x + \frac{\pi}{4}\right)\right]$, $0 < x \le 2\pi$ (where, [.] denotes the greatest integer $\le x$). The number of points of discontinuity of $f(x)$ are

(a) 3 (b) 4 (c) 5 (d) 6

32 If $f(x) = \tan^{-1}\left[\frac{\sin x}{1 + \cos x}\right]$, then $f'\left[\frac{\pi}{3}\right]$ is equal to

(a) $\frac{1}{2(1 + \cos x)}$ (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) None of these

33 Let $f(x)$ be a polynomial function of the second degree. If $f(1) = f(-1)$ and a_1, a_2, a_3 are in AP, then $f'(a_1), f'(a_2)$ and $f'(a_3)$ are in

(a) AP (b) GP

(c) HP (d) None of these

34 If $y = \cos^{-1}(\cos x)$, then $y'(x)$ is equal to

(a) 1 for all x

(b) -1 for all x

(c) 1 in 2nd and 3rd quadrants

(d) -1 in 3rd and 4th quadrants

35 In $(-4, 4)$ the function $f(x) = \int_{-10}^{x} (t^4 - 4)\, e^{-4t} dt$ has

(a) no extreme (b) one extreme
(c) two extreme (d) four extreme

36 If $f(x) = kx^3 - 9x^2 + 9x + 3$ is monotonically increasing in each interval, then

(a) $k < 3$ (b) $k \le 3$
(c) $k > 3$ (d) None of these

37 The integral

$$\int_{-1}^{3} \left(\tan^{-1} \frac{x}{x^2+1} + \tan^{-1} \frac{x^2+1}{x} \right) dx \text{ is equal to}$$

(a) π (b) 2π
(c) 3π (d) None of these

38 $\lim_{n \to \infty} \dfrac{1^{99} + 2^{99} + 3^{99} + \ldots + n^{99}}{n^{100}}$ is equal to

(a) $\dfrac{9}{100}$ (b) $\dfrac{1}{100}$ (c) $\dfrac{1}{99}$ (d) $\dfrac{1}{101}$

39 If $\int (\sin 2x + \cos 2x)\, dx = \dfrac{1}{\sqrt{2}} \sin(2x - c) + a$, then the value of a and c is

(a) $c = \dfrac{\pi}{4}$ and $a = k$

(b) $c = -\dfrac{\pi}{4}$ and $a = \dfrac{\pi}{2}$

(c) $c = \dfrac{\pi}{2}$ and a is an arbitrary constant

(d) None of the above

40 $\int \dfrac{\cos 2x}{(\cos x + \sin x)^2} dx$ is equal to

(a) $\log \sqrt{\cos x + \sin x} + C$ (b) $\log(\cos x - \sin x) + C$
(c) $\log(\cos x + \sin x) + C$ (d) $-\dfrac{1}{\cos x + \sin x} + C$

41 If $\int \dfrac{\cos 4x + 1}{\cot x - \tan x} dx = k \cos 4x + C$, then

(a) $k = -\dfrac{1}{2}$ (b) $k = -\dfrac{1}{8}$
(c) $k = -\dfrac{1}{4}$ (d) None of these

42 $\int \dfrac{dx}{\sin(x-a)\sin(x-b)}$ is equal to

(a) $\dfrac{1}{\sin(a-b)} \log \left| \dfrac{\sin(x-a)}{\sin(x-b)} \right| + C$

(b) $\dfrac{-1}{\sin(a-b)} \log \left| \dfrac{\sin(x-a)}{\sin(x-b)} \right| + C$

(c) $\log \sin(x-a) \cdot \sin(x-b) + C$

(d) $\log \left| \dfrac{\sin(x-a)}{\sin(x-b)} \right| + C$

43 $\int e^x \dfrac{(x^2+1)}{(x+1)^2} dx$ is equal to

(a) $\left(\dfrac{x-1}{x+1} \right) e^x + C$ (b) $e^x \left(\dfrac{x+1}{x-1} \right) + C$
(c) $e^x (x+1)(x-1) + C$ (d) None of these

44 $I_1 = \int \sin^{-1} x\, dx$ and $I_2 = \int \sin^{-1} \sqrt{1 - x^2}\, dx$, then

(a) $I_1 = I_2$ (b) $I_2 = \dfrac{\pi}{2 I_1}$

(c) $I_1 + I_2 = \dfrac{\pi x}{2}$ (d) $I_1 + I_2 = \dfrac{\pi}{2}$

45 $\int_0^{\pi/4} \dfrac{dx}{\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x}$ is equal to

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$

(c) $\dfrac{\pi}{3}$ (d) None of these

46 If $f(x) = |x-1| + |x-3| + |5-x|, \forall x \in R$. If $f(x)$ is increases, then x belongs to

(a) $(1, \infty)$ (b) $(3, \infty)$ (c) $(5, \infty)$ (d) $(1, 3)$

47 The value of $\int \dfrac{dx}{(1+x)^{1/2} - (1+x)^{1/3}}$ is

(a) $2\lambda^{1/2} + 3\lambda^{1/3} + 6\lambda^{1/6} + 6 \ln |\lambda^{1/6} - 1| + C$
(b) $2\lambda^{1/2} - 3\lambda^{1/3} + 6\lambda^{1/6} + 6 \ln |\lambda^{1/6} - 1| + C$
(c) $2\lambda^{1/2} + 3\lambda^{1/3} - 6\lambda^{1/6} + 6 \ln |\lambda^{1/6} - 1| + C$
(d) $2\lambda^{1/2} + 3\lambda^{1/3} + 6\lambda^{1/6} - 6 \ln |\lambda^{1/6} - 1| + C$
(where, $\lambda = 1 + x$)

48 The value of $\int \dfrac{\sqrt[4]{x}}{\sqrt{x} - 1} dx$ is

(a) $-\dfrac{4}{3} x^{3/4} + 4x^{1/4} + 2 \ln \left| \dfrac{x^{1/4} - 1}{x^{1/4} + 1} \right| + C$

(b) $\dfrac{4}{3} x^{3/4} + 4x^{1/4} + 2 \ln \left| \dfrac{x^{1/4} - 1}{x^{1/4} + 1} \right| + C$

(c) $-\dfrac{4}{3} x^{3/4} - 4x^{1/4} + 2 \ln \left| \dfrac{x^{1/4} - 1}{x^{1/4} + 1} \right| + C$

(d) $\dfrac{4}{3} x^{3/4} - 4x^{1/4} + 2 \ln \left| \dfrac{x^{1/4} - 1}{x^{1/4} + 1} \right| + C$

49 The value of n for which the function

$$f(x) = \begin{cases} \dfrac{((5)^x - 1)^3}{\sin\left(\dfrac{x}{n}\right) \cdot \log\left(1 + \dfrac{x^2}{3}\right)}, & x \ne 0 \\ 15 (\log 5)^3, & x = 0 \end{cases}$$

may be continuous at $x = 0$, is

(a) 5 (b) 3 (c) 4 (d) 2

50 The longest distance of the point $(4, 0)$ from the curve $2x(1-x) = y^2$ is equal to

(a) 3 units (b) 4.5 units
(c) 5 units (d) None of these

51 The normal to the curve $5x^5 - 10x^3 + x + 2y + 4 = 0$ at $P(0, -2)$ meets the curve again at two points at which equation of tangents to the curve is equal to

(a) $y = 3x + 2$ (b) $y = 2(x-1)$
(c) $3y + 2x + 7 = 0$ (d) None of these

52 $\int_{-1}^{2} \{|x-1| + [x]\}\, dx$, where $[x]$ is greatest integer is equal to

(a) 8 (b) 9 (c) $\frac{5}{2}$ (d) 4

53 The equation of the curve passing through the point $\left(\frac{1}{2}, \frac{\pi}{8}\right)$ and having slope of tangent at any point (x, y) as $(y/x) - \cos^2(y/x)$ is equal to

(a) $y = x\tan^{-1}\left(\log \frac{e}{2x}\right)$ (b) $x = y\tan^{-1}\left(\frac{e}{x^2}\right)$

(c) $y = x^2 \tan^{-1}\left(\frac{2x}{e}\right)$ (d) None of these

54 If $x\, dy = y\,(dx + y\, dy)$, $y(1) = 1$ and $y(x) > 0$, then $y(-3)$ is equal to

(a) 3 (b) 2 (c) 1 (d) 0

55 $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, dx$ is equal to

(a) $\frac{1}{20}\log 3$ (b) $\log 3$

(c) $\frac{1}{20}\log 5$ (d) None of these

56 If $f(x) = \begin{cases} e^{\cos x} \sin x, & |x| \le 2 \\ 2, & \text{otherwise} \end{cases}$, then $\int_{-2}^{3} f(x)\, dx$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

57 The area bounded by curves $y = \cos x$ and $y = \sin x$ and ordinates $x = 0$ and $x = \frac{\pi}{4}$ is

(a) $\sqrt{2}$ (b) $\sqrt{2} + 1$ (c) $\sqrt{2} - 1$ (d) $\sqrt{2}(\sqrt{2} - 1)$

Direction (Q. Nos. 58-66) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true; Statement II is false

(d) Statement I is false; Statement II is true

58 **Statement I** The tangents to curve $y = 7x^3 + 11$ at the points, where $x = 2$ and $x = -2$ are parallel.

Statement II The slope of the tangents at the points, where $x = 2$ and $x = -2$, are equal.

59 **Statement I** The derivative of

$$f(x) = \int_{1/x}^{\sqrt{x}} \cos t^2 dt, (x > 0) \text{ at } x = 1 \text{ is } \left(\frac{3}{2}\right)\cos 1.$$

Statement II $\frac{d}{dx}\int_{\psi(x)}^{\phi(x)} f(t)\, dt = f(\phi(x)) - f(\psi(x))$

60 **Statement I** The solution of the equation

$$x\frac{dy}{dx} + 6y = 3xy^{4/3} \text{ is } y(x) = \frac{1}{(x + cx^2)^3}.$$

Statement II The solution of a linear equation is obtained by multiplying with its integrating factor.

61 **Statement I** $\int \frac{x^2 - 1}{(x^2 + 1)\sqrt{x^4 + 1}}\, dx = \sec^{-1}\left|\frac{x^2 + 1}{x\sqrt{2}}\right| + C$

Statement II $\int \frac{dt}{t\sqrt{t^2 - a}} = \frac{1}{\sqrt{a}}\sec^{-1}\left|\frac{t}{\sqrt{a}}\right| + C$

62 **Statement I** The absolute minimum value of $|x - 1| + |x - 2| + |x - 3|$ is 2.

Statement II The function $|x - 1| + |x - 2| + |x - 3|$ is differentiable on $R \sim \{1, 2\}$.

63 **Statement I** If $f(x) = \max\{x^2 - 2x + 2, |x - 1|\}$, then the greatest value of $f(x)$ on the interval [0, 3] is 5.

Statement II Greatest value, $f(3) = \max\{5, 2\} = 5$

64 **Statement I** The point of contact of the vertical tangents to $x = 2 - 3\sin\theta$, $y = 3 + 2\cos\theta$ are $(-1, 3)$ and $(5, 3)$.

Statement II For vertical tangent, $dx/d\theta = 0$

65 Let $f : R \to R$ be differentiable and strictly increasing function throughout its domain.

Statement I If $|f(x)|$ is also strictly increasing function, then $f(x) = 0$ has no real roots.

Statement II At ∞ or $-\infty$ $f(x)$ may approach to 0, but cannot be equal to zero.

66 **Statement I** The area by region $|x + y| + |x - y| \le 2$ is 4 sq units.

Statement II Area enclosed by region $|x + y| + |x - y| \le 2$ is symmetric about *X*-axis.

ANSWERS

1 (d)	**2** (d)	**3** (b)	**4** (d)	**5** (d)	**6** (a)	**7** (b)	**8** (a)	**9** (c)	**10** (c)
11 (b)	**12** (c)	**13** (c)	**14** (c)	**15** (b)	**16** (b)	**17** (d)	**18** (b)	**19** (b)	**20** (a)
21 (a)	**22** (c)	**23** (a)	**24** (a)	**25** (c)	**26** (c)	**27** (c)	**28** (a)	**29** (c)	**30** (c)
31 (d)	**32** (b)	**33** (a)	**34** (d)	**35** (c)	**36** (c)	**37** (b)	**38** (b)	**39** (a)	**40** (c)
41 (b)	**42** (a)	**43** (a)	**44** (c)	**45** (a)	**46** (b)	**47** (a)	**48** (b)	**49** (a)	**50** (c)
51 (b)	**52** (c)	**53** (a)	**54** (a)	**55** (a)	**56** (c)	**57** (c)	**58** (a)	**59** (c)	**60** (b)
61 (d)	**62** (c)	**63** (b)	**64** (a)	**65** (b)	**66** (b)				

Hints and Explanations

1 Given, $f:(2,3)\to(0,1)$

and $f(x)=x-[x]$

$\therefore \quad f(x)=y=x-2$

$\Rightarrow \quad x=y+2=f^{-1}(y)$

$\Rightarrow \ f^{-1}(x)=x+2$

2 Let

$$I=\int \frac{dx}{\sqrt{2}\left(\sin x\cdot\sin\frac{\pi}{4}-\cos x\cdot\cos\frac{\pi}{4}+1\right)}$$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{1-\cos 2\left(\frac{x}{2}+\frac{\pi}{8}\right)}$$

$$=\frac{1}{2\sqrt{2}}\int \operatorname{cosec}^2\left(\frac{x}{2}+\frac{\pi}{8}\right)dx$$

$$=\frac{1}{2\sqrt{2}}\,\frac{-\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)}{1/2}+C$$

$$=-\frac{1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$$

3 $$\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}$$

$$=\lim_{n\to\infty}\left(\frac{n(n-1)+1}{n(n-1)-1}\right)^{n(n-1)}$$

$$=\lim_{n\to\infty}\frac{\left\{1+\frac{1}{n(n-1)}\right\}^{n(n-1)}}{\left\{1-\frac{1}{n(n-1)}\right\}^{n(n-1)}}=\frac{e}{e^{-1}}=e^2$$

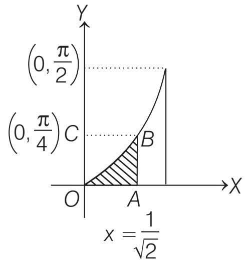
4 Slope of the normal $=\dfrac{-1}{\frac{dy}{dx}}$

$$\Rightarrow \tan\frac{3\pi}{4}=\frac{-1}{\left(\frac{dy}{dx}\right)_{(3,\,4)}}\Rightarrow\left(\frac{dy}{dx}\right)_{(3,\,4)}=1$$

$\therefore \qquad f'(3)=1$

5 Required area

= Area of rectangle $OABC$ – Area of curve $OBCO$



$$=\frac{\pi}{4\sqrt{2}}-\int_0^{\pi/4}\sin y\,dy$$

$$=\frac{\pi}{4\sqrt{2}}+[\cos y]_0^{\pi/4}$$

$$=\left(\frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right)\text{ sq units}$$

6 $$f(-\theta)=\log\left(\frac{2-\sin\theta}{2+\sin\theta}\right)^{-1}$$

$$=-\log\left(\frac{2-\sin\theta}{2+\sin\theta}\right)=-f(\theta)$$

$f(\theta)$ is an odd function of θ.

$$\therefore\quad I=\int_{-\pi/2}^{\pi/2}\log\left(\frac{2-\sin\theta}{2+\sin\theta}\right)d\theta=0$$

7. Given,

$(2x-y+1)\,dx+(2y-x+1)\,dy=0$

$$\Rightarrow\quad\frac{dy}{dx}=\frac{2x-y+1}{x-2y-1},$$

Put $\quad x=X+h,\ y=Y+k$

$$\Rightarrow\quad\frac{dY}{dX}=\frac{2X-Y+2h-k+1}{X-2Y+h-2k-1}$$

Again, put $2h-k+1=0$

and $\quad h-2k-1=0$

On solving,

$h=-1,\ k=-1$

$$\therefore\quad\frac{dY}{dX}=\frac{2X-Y}{X-2Y}$$

On putting $Y=vX$, we get

$$\Rightarrow\quad v+X\frac{dv}{dX}=\frac{2X-vX}{X-2vX}=\frac{2-v}{1-2v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{2-2v+2v^2}{1-2v}=\frac{2(v^2-v+1)}{1-2v}$$

$$\therefore\quad\frac{dX}{X}=\frac{(1-2v)}{2(v^2-v+1)}dv$$

Put $\quad v^2-v+1=t$

$\Rightarrow\quad(2v-1)\,dv=dt$

$$\therefore\quad\frac{dX}{X}=-\frac{dt}{2t}$$

On integrating,

$\log X=\log t^{-1/2}+\log C^{1/2}$

$\therefore\quad X=t^{-1/2}C^{1/2}$

$\Rightarrow\quad X=(v^2-v+1)^{-1/2}C^{1/2}$

$\Rightarrow\quad X^2(v^2-v+1)=C$

$$\Rightarrow(x+1)^2\left[\frac{(y+1)^2}{(x+1)^2}-\frac{(y+1)}{(x+1)}+1\right]=C$$

$$\Rightarrow(y+1)^2-(y+1)(x+1)+(x+1)^2=C$$

$\Rightarrow\quad y^2+x^2-xy+x+y=C-1$

$\Rightarrow\quad x^2+y^2-xy+x+y=C$

8 $f(x)=x^4-\dfrac{x^3}{3}$

$\Rightarrow f'(x)=4x^3-x^2$

For increasing,

$4x^3-x^2>0\Rightarrow x^2(4x-1)>0$

Therefore, the function is increasing for $x>1/4$.

Similarly, decreasing for $x<\dfrac{1}{4}$.

9 Given,

$$y=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}$$

$$\Rightarrow\frac{dy}{dx}=0+1+x+\frac{x^2}{2!}+\ldots+\frac{x^{n-1}}{(n-1)!}$$

$$\Rightarrow\frac{dy}{dx}+\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\ldots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}$$

$$\Rightarrow\quad\frac{dy}{dx}=y-\frac{x^n}{n!}$$

10 $$\lim_{x\to1}\frac{x^{1/4}-x^{1/5}}{x^3-1}=\lim_{x\to1}\frac{\frac{1}{4}x^{-3/4}-\frac{1}{5}x^{-4/5}}{3x^2}$$

$$=\frac{\frac{1}{4}-\frac{1}{5}}{3}=\frac{1}{60}$$

11 Given, $f(x)=|x-1|+|x-3|$

$$f(x)=\begin{cases}-(x-1)-(x-3), & x<1\\(x-1)-(x-3), & 1\le x<3\\(x-1)+(x-3), & x>3\end{cases}$$

$$=\begin{cases}4-2x, & x<1\\2, & 1<x<3\\2x-4, & x\ge3\end{cases}$$

In the neighbourhood of $x=2$, $f(x)=2$

Hence, $f'(x)=0$

12 The given function is periodic with period 2π. So, the difference between the greatest and least values of the function is the difference between these values on the interval $[0,2\pi]$.

Now,

$f'(x)=-(\sin x+\sin 2x-\sin 3x)$

$=-4\sin x\sin(3x/2)\sin(x/2)$

Hence, $x = 0, 2\pi/3, \pi$ and 2π are the critical points.

Also, $f(0) = 1 + \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$

$$f\left(\frac{2\pi}{3}\right) = -\frac{13}{12}, f(\pi) = -\frac{1}{6}$$

and $f(2\pi) = \frac{7}{6}$

$\therefore$ Required difference

$$= \frac{7}{6} - \left(-\frac{13}{12}\right) = \frac{27}{12} = \frac{9}{4}$$

13 From mean value theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Given, $a = 0 \Rightarrow f(a) = 0$

and $b = \frac{1}{2} \Rightarrow f(b) = \frac{3}{8}$

$$f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$$

$$f'(c) = (c-1)(c-2) + c(c-2) + c(c-1)$$

$$= c^2 - 3c + 2 + c^2 - 2c + c^2 - c$$

$\Rightarrow f'(c) = 3c^2 - 6c + 2$

According to mean value theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{(3/8) - 0}{(1/2) - 0} = \frac{3}{4}$$

$$\Rightarrow 3c^2 - 6c + \frac{5}{4} = 0$$

$$\Rightarrow c = \frac{6 \pm \sqrt{36 - 15}}{2 \times 3} = \frac{6 \pm \sqrt{21}}{6}$$

$$= 1 \pm \frac{\sqrt{21}}{6} = 1 - \frac{\sqrt{21}}{6}$$

$$\left[\because \left(1 + \frac{\sqrt{21}}{6}\right) \notin \left(0, \frac{1}{2}\right)\right]$$

14 We know that, in a triangle

$$\angle A + \angle B + \angle C = \pi$$

$\therefore \quad dA + dB + dC = 0$

If R is circumradius, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$a = 2R \sin A$$

On differentiating, we get

$$da = 2R \cos A \, dA$$

$$\frac{da}{\cos A} = 2R\,dA$$

Similarly, $\frac{db}{\cos B} = 2R\,dB$

and $\frac{dc}{\cos C} = 2R\,dC$

$$\therefore \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R(dA + dB + dC) = 0$$

15 $\therefore \int_0^{1.5} [x^2]\,dx$

$$= \int_0^1 [x^2]\,dx + \int_1^{\sqrt{2}} [x^2]\,dx + \int_{\sqrt{2}}^{1.5} [x^2]\,dx$$

$$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{1.5} 2\,dx$$

$$= 0 + (x)_1^{\sqrt{2}} + 2(x)_{\sqrt{2}}^{1.5}$$

$$= \sqrt{2} - 1 + 3 - 2\sqrt{2} = 2 - \sqrt{2}$$

16 The differential equation is

$$\frac{dy}{dx} - y \tan x = -y^2 \sec x$$

This is Bernoulli's equation, which can be reducible to linear equation.

On dividing the equation by y^2, we get

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x \quad \text{...(i)}$$

Put $\frac{1}{y} = Y \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dY}{dx}$

Eq. (i) reduces to

$$-\frac{dY}{dx} - Y \tan x = -\sec x$$

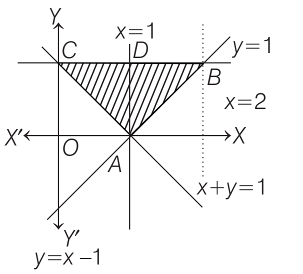
$\Rightarrow \frac{dY}{dx} + Y \tan x = \sec x$, which is a linear equation.

Hence, IF $= e^{\int \tan x\,dx} = \sec x$

17 $A = \int_0^2 [2^x - (2x - x^2)]\,dx$

$$= \left[\frac{2^x}{\log 2} - x^2 + \frac{x^3}{3}\right]_0^2 = \frac{3}{\log 2} - \frac{4}{3}$$

18 $y = x - 1$, if $x > 1$ and $y = -(x-1)$; if $x < 1$



Area of bounded region,

A = Area of ΔABC

$$= \frac{1}{2} \times BC \times AD = \frac{1}{2} \times 2 \times 1$$

$$= 1 \text{ sq unit}$$

19 $I_1 = \int_{1-k}^{k} (1 - x) \sin[x(1 - x)]\,dx$

[by property]

$$= \int_{1-k}^{k} \sin x(1 - x)\,dx - \int_{1-k}^{k} x \sin x(1 - x)\,dx$$

$$= I_2 - I_1 \text{ or } 2I_1 = I_2$$

20 $I_1 = \int_{\sec^2 z}^{2 - \tan^2 z} (3 - x) f(x(3 - x))\,dx$

[by property]

$$I_1 = 3\int_{\sec^2 z}^{2 - \tan^2 z} f(x(3 - x))\,dx - \int_{\sec^2 z}^{2 - \tan^2 z} x f(x(3 - x))\,dx$$

$$I_1 = 3I_2 - I_1 \Rightarrow \frac{I_1}{I_2} = \frac{3}{2}$$

21 Let $I = \int_0^{\pi} [f(x) + g(x)]\,dx$

$$I = \int_0^{\pi} [f(\pi - x) + g(\pi - x)]\,dx$$

$$= \int_0^{\pi} [1 - f(x) + 1 - g(x)]\,dx$$

$$= 2\int_0^{\pi} dx - \int_0^{\pi} [f(x) + g(x)]\,dx$$

$\Rightarrow 2I = 2\pi$

$\therefore \quad I = \pi$

22 At $x = 1$, $f(x) = 1$,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} |2x - 3|[x] = 1$$

and $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sin\frac{\pi x}{2} = 1$

Hence, $f(x)$ is continuous at $x = 1$.

Now,

$$\lim_{h \to 0^+} \frac{f(1+h) - 1}{h} = \lim_{h \to 0^+} \frac{|2h - 1| - 1}{h}$$

$$= \lim_{h \to 0^+} \frac{1 - 2h - 1}{h} = -2$$

and $\lim_{h \to 0^-} \frac{f(1+h) - 1}{h}$

$$= \lim_{h \to 0^-} \frac{\sin\left(\frac{\pi}{2} + \frac{\pi h}{2}\right) - 1}{h}$$

$$= \lim_{h \to 0^-} \frac{\cos\frac{\pi h}{2} - 1}{h} = 0$$

Hence, $f(x)$ is not differentiable at $x = 1$.

23 Given that,

$$y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots \infty}}}$$

$$y = \sqrt{f(x) + y}$$

On squaring both sides, we get

$$y^2 - y = f(x)$$

On differentiating w.r.t. x, we get

$$(2y - 1)\frac{dy}{dx} = f'(x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{f'(x)}{2y - 1}$$

24 We have,

$h(x) = f(x) - [f(x)]^2 + [f(x)]^3$

$h'(x) = f'(x) - 2f(x)\,f'(x) + 3\,[f(x)]^2 \cdot f'(x)$

$= f'(x)\,[1 - 2f(x) + 3\,[f(x)]^2]$

$= 3f'(x)\left\{\left(f(x) - \frac{1}{3}\right)^2 + \frac{2}{9}\right\}$

Hence, $h'(x)$ and $f'(x)$ have same sign.

25 We have,

$u = \int e^{ax}\cos bx\,dx = e^{ax}\cdot\frac{\sin bx}{b} - \frac{a}{b}\int e^{ax}\cdot\sin bx\,dx$

$= \frac{e^{ax}\cdot\sin bx}{b} - \frac{a}{b}v$

$\Rightarrow bu + av = e^{ax}\cdot\sin bx$ ...(i)

Similarly, $bv - au = -e^{ax}\cdot\cos bx$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$(a^2 + b^2)(u^2 + v^2) = e^{2ax}$

26 Let $I = \int_a^b f(x)\cdot g(x)\,dx$

Put $f(x) = t \Rightarrow f'(x)\,dx = dt$

$\Rightarrow g(x)\,dx = dt$

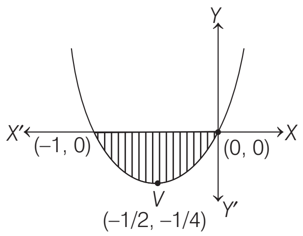
$I = \int_{f(a)}^{f(b)} t\,dt = \left|\frac{t^2}{2}\right|_{f(a)}^{f(b)} = \frac{[f(b)]^2 - [f(a)]^2}{2}$

27 Given, $\frac{dy}{dx} = 2x + 1 \Rightarrow y = x^2 + x + C$

$\Rightarrow y = x^2 + x,$

[$\because C = 0$ by putting $x = 1, y = 2$]

$\Rightarrow \left(x + \frac{1}{2}\right)^2 = y + \frac{1}{4}$, which is an equation of parabola whose vertex is $V\left(\frac{-1}{2}, \frac{-1}{4}\right)$.



$\therefore$ Required area

$= \left|\int_{-1}^{0}(x^2 + x)\,dx\right| = \left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{-1}^{0}$

$= \left|\frac{-1}{3} + \frac{1}{2}\right| = \frac{1}{6}$ sq unit

28 If $x \in R - (0, 1)$, then

$x \le x^2 \;\Rightarrow x(1 - x) \le 0$

$\Rightarrow x(x - 1) \ge 0 \Rightarrow x \le 0$ or $x \ge 1$,

$\therefore\; h(x) = \begin{bmatrix} x, & x \le 0 \\ x^2, & 0 < x < 1 \\ x, & x \ge 1 \end{bmatrix}$

$h(x)$ is continuous for everywhere but not differentiable at $x = 0$ and 1.

i.e.

$h'(x) = \begin{bmatrix} 1, & x < 0 \\ 2x, & 0 < x < 1 \\ \text{not exist}, & x = 0 \\ 1, & x > 1 \\ \text{not exist}, & x = 1 \end{bmatrix}$

$\therefore\; h'(x) = 1, \forall\, x > 1$

29 Since, f is continuous at $x = 0$.

$\therefore\; f(0^-) = f(0^+) = f(0) = -1$

Also, f is differentiable at $x = 0$, therefore $Lf'(0) = Rf'(0)$

$\Rightarrow \lim_{h\to 0}\frac{f(0 - h) - f(0)}{-h} = \lim_{h\to 0}\frac{f(0 + h) - f(0)}{h}$

$\Rightarrow \lim_{h\to 0}\left(\frac{e^{-2h} - 1 + 1}{-h}\right) = \lim_{h\to 0}\frac{\left(ah + \frac{bh^2}{2} - 1 + 1\right)}{h}$

$\Rightarrow \lim_{h\to 0}\left(\frac{-2e^{-2h}}{-1}\right) = \lim_{h\to 0}(a + bh)$

[L' Hospital's rule]

$\Rightarrow 2 = a + 0 \Rightarrow a = 2, b$ any number

30 $y = \lim_{x\to\pi/2}\frac{\int_{\pi/2}^{x} t\,dt}{\sin(2x - \pi)}$

$\Rightarrow y = \lim_{x\to\pi/2}\frac{\left[\frac{t^2}{2}\right]_{\pi/2}^{x}}{\sin(2x - \pi)}$

$\Rightarrow y = \lim_{x\to\pi/2}\frac{\left(\frac{x^2}{2} - \frac{\pi^2}{8}\right)}{\sin(2x - \pi)}$

$\Rightarrow y = \lim_{x\to\pi/2}\frac{1}{8}\frac{(4x^2 - \pi^2)}{\sin(2x - \pi)}$

$\Rightarrow y = \lim_{x\to\pi/2}\frac{1}{8}\frac{(2x - \pi)(2x + \pi)}{\sin(2x - \pi)}$

$\Rightarrow y = \frac{1}{8}\frac{\lim_{x\to\pi/2}(2x + \pi)}{\lim_{x\to\pi/2}\frac{\sin(2x - \pi)}{2x - \pi}}$

$\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$

$\Rightarrow y = \frac{1}{8}\times 2\pi = \frac{1}{4}\pi$

31 Using the fact that $[x]$ is discontinuous at all integer numbers.

$\therefore\; f(x) = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$ is an integer for

$x + \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \frac{9\pi}{4}$

$\Rightarrow x = \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, 2\pi$

32 $f(x) = \tan^{-1}\left[\frac{\sin x}{1 + \cos x}\right]$

$= \tan^{-1}\left[\tan\frac{x}{2}\right] = \frac{x}{2} \Rightarrow f'(x) = \frac{1}{2}$

Hence, $f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$

33 Let $f(x) = ax^2 + bx + c$

Then, $f'(x) = 2ax + b$

Also, $f(1) = f(-1)$

$\Rightarrow a + b + c = a - b + c$

$\Rightarrow b = 0$

$\therefore\; f'(x) = 2ax;$

$\Rightarrow f'(a_1) = 2aa_1,\; f'(a_2) = 2aa_2,\; f'(a_3) = 2aa_3$

As a_1, a_2, a_3 are in AP,

$f'(a_1), f'(a_2), f'(a_3)$ are in AP.

34 Given, $y = \cos^{-1}(\cos x)$

$\Rightarrow y'(x) = \frac{1}{\sqrt{1 - \cos^2 x}}\sin x = \frac{\sin x}{|\sin x|}$

$= \begin{cases} 1, & \text{1st and 2nd quadrants} \\ -1, & \text{3rd and 4th quadrants} \end{cases}$

35 $f(x) = \int_{-10}^{x}(t^4 - 4)e^{-4t}dt$

$\Rightarrow f'(x) = (x^4 - 4)e^{-4x}$

Now, $f'(x) = 0 \;\Rightarrow x = \pm\sqrt{2}$

Now,

$f''(x) = -4(x^4 - 4)e^{-4x} + 4x^3e^{-4x}$

At $x = \sqrt{2}$ and $x = -\sqrt{2}$, the given function has extreme value.

36 $f'(x) = 3kx^2 - 18x + 9$

$= 3[kx^2 - 6x + 3] > 0, \forall x \in R$

$\therefore\; \Delta = b^2 - 4ac < 0, k > 0$

i.e. $36 - 12k < 0$ or $k > 3$.

37 Let $I = \int_{-1}^{3}\left\{\tan^{-1}\left(\frac{x}{x^2 + 1}\right) + \tan^{-1}\left(\frac{x^2 + 1}{x}\right)\right\}dx$

$$= \int_{-1}^{3} \left\{\tan^{-1}\left(\frac{x}{x^2+1}\right) + \cot^{-1}\left(\frac{x}{x^2+1}\right)\right\} dx$$

$$= \int_{-1}^{3} \frac{\pi}{2}\, dx = 2\pi$$

$$\left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \forall\ x \in R\right]$$

38 $\lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{r^{99}}{n^{100}}\right) = \int_0^1 x^{99} dx$

$$= \left[\frac{x^{100}}{100}\right]_0^1 = \frac{1}{100}$$

39 $I = \frac{-\cos 2x}{2} + \frac{\sin 2x}{2} + k$

$$= \frac{1}{\sqrt{2}}\left(\sin 2x \cos\frac{\pi}{4} - \cos 2x \sin\frac{\pi}{4}\right) + k$$

$$= \frac{1}{\sqrt{2}} \sin\left(2x - \frac{\pi}{4}\right) + k$$

$\therefore \quad c = \frac{\pi}{4};\ a = k$

40 $\int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2} dx$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

Put $\sin x + \cos x = t$

$\Rightarrow (\cos x - \sin x)\, dx = dt$

$\therefore \quad \int \frac{1}{t}\, dt = \log t + C$

$$= \log(\sin x + \cos x) + C$$

41 $\int \frac{2\cos^2 2x}{\cos^2 x - \sin^2 x} \cdot \sin x \cos x\, dx$

$$= \int \cos 2x \cdot \sin 2x\, dx$$

$$= \frac{1}{2}\int \sin 4x\, dx = -\frac{1}{8}\cos 4x + C$$

$\therefore \quad k = -\frac{1}{8}$

42 $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \left[\int \cot(x-a)\, dx - \int \cot(x-b)\, dx\right]$$

$$= \frac{1}{\sin(a-b)} \log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$$

43 $\int \frac{e^x (x^2 - 1 + 2)}{(x+1)^2} dx$

$$= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right] dx$$

$$= \int e^x\, [f(x) + f'(x)]\, dx$$

$$\left[\because f(x) = \frac{x-1}{x+1}, f'(x) = \frac{2}{(x+1)^2}\right]$$

$$= e^x\left(\frac{x-1}{x+1}\right) + C$$

44 $I_1 = \int \sin^{-1} x\, dx.$

Let $\sin^{-1} x = \theta \Rightarrow x = \sin\theta$

$\Rightarrow \quad dx = \cos\theta\, d\theta$

$\therefore \quad I_1 = \int \theta \cos\theta\, d\theta$

$= \theta\sin\theta - \int \sin\theta\, d\theta = \theta\sin\theta + \cos\theta$

$= x\sin^{-1} x + \sqrt{1-x^2}$

and $\quad I_2 = \int \sin^{-1} \sqrt{1-x^2}\, dx$

$= \int \cos^{-1} x\, dx$

Let $\cos\phi = x$

$\Rightarrow -\sin\phi\, d\phi = dx$

$\therefore\ I_2 = -\int \phi\sin\phi\, d\phi = \phi\cos\phi + \int -\cos\phi\, d\phi$

$= \phi\cos\phi - \sin\phi$

$= x\cos^{-1} x - \sqrt{1-x^2}$

$\therefore \quad I_1 + I_2 = x(\cos^{-1} x + \sin^{-1} x)$

$= \frac{\pi x}{2}$

45 Divide numerator and denominator by $\cos^4 x$,

$\therefore \quad I = \int_0^{\pi/4} \frac{\sec^2 x \sec^2 x\, dx}{1 - \tan^2 x + \tan^4 x}$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore \quad I = \int_0^1 \frac{1+t^2}{t^4 - t^2 + 1} dt$

$$= \int_0^1 \frac{1 + \frac{1}{t^2}}{t^2 - 1 + \frac{1}{t^2}} dt = \int_0^1 \frac{\left(1+\frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 1} dt$$

Put $\quad z = t - \frac{1}{t}$

and $\quad dz = \left(1 + \frac{1}{t^2}\right) dt$

$$I = \int_{-\infty}^{0} \frac{dz}{1+z^2} dz = [\tan^{-1} z]_{-\infty}^{0}$$

$$= \tan^{-1}(0) - \tan^{-1}(-\infty) = \frac{\pi}{2}$$

46 Given,

$f(x) = |x-1| + |x-3| + |5-x|,$

$\forall\ x \in R$

$$\therefore \quad f(x) = \begin{cases} 9 - 3x, & x < 1 \\ 7 - x, & 1 \le x < 3 \\ x + 1, & 3 \le x < 5 \\ 3x - 9, & x \ge 5 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} -3, & x < 1 \\ -1, & 1 < x < 3 \\ 1, & 3 < x < 5 \\ 3, & x > 5 \end{cases}$$

It is clear that $f'(x) > 0$, when $x \in (3, \infty)$.

47 Let $I = \int \frac{dx}{(1+x)^{1/2} - (1+x)^{1/3}}$

Put $1 + x = t^6 \quad \Rightarrow \quad dx = 6t^5\, dt$

Then, $\quad I = \int \frac{6t^5 dt}{(t^3 - t^2)} = 6\int \frac{t^3}{(t-1)} dt$

$$= 6\int \frac{(t^3-1)+1}{(t-1)} dt$$

$$= 6\int \left(t^2 + t + 1 + \frac{1}{t-1}\right) dt$$

$$= 6\left(\frac{t^3}{3} + \frac{t^2}{2} + t + \ln|t-1|\right) + C$$

$= 2(1+x)^{1/2} + 3(1+x)^{1/3} + 6(1+x)^{1/6} + 6\ln|(1+x)^{1/6} - 1| + C$

$= 2\lambda^{1/2} + 3\lambda^{1/3} + 6\lambda^{1/6} + 6\ln|\lambda^{1/6} - 1| + C \quad$ [where, $\lambda = 1 + x$]

48 Let $\quad I = \int \frac{x^{1/4}}{x^{1/2} - 1} dt$

Put $\quad x = t^4 \Rightarrow dx = 4t^3 dt$

$$\therefore I = \int \frac{t \cdot 4t^3 dt}{(t^2 - 1)} = 4\int \left(\frac{t^4 - 1 + 1}{t^2 - 1}\right) dt$$

$$= 4\int \left(t^2 + 1 + \frac{1}{t^2 - 1}\right) dt$$

$$= 4\left\{\frac{t^3}{3} + t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right\} + C$$

$$= \frac{4}{3} x^{3/4} + 4x^{1/4} + 2\ln\left|\frac{x^{1/4} - 1}{x^{1/4} + 1}\right| + C$$

49 Clearly,

$$= \lim_{x\to 0} \frac{(5^x - 1)^3}{\sin\left(\frac{x}{n}\right) \cdot \log\left(1 + \frac{x^2}{3}\right)}$$

$$= \lim_{x\to 0} \frac{(5^x - 1)^3}{\frac{x}{n}\left(\frac{x^2}{3} + \dots\right)} \cdot \lim_{x\to 0} \frac{1}{\frac{\sin x/n}{x/n}}$$

$$= \lim_{x\to 0} \left\{\frac{(5^x - 1)}{x}\right\}^3 \cdot 3n$$

$= 3n(\log 5)^3 \quad \dots$(i)

Since, the value of the function at $x = 0$ is $15(\log 5)^3$.

$\therefore \quad 3n(\log 5)^3 = 15(\log 5)^3$

$\Rightarrow \quad n = 5$

50 Let the distance of point $(4, 0)$ from the point (x, y) lying on the curve be

$D^2 = (x-4)^2 + y^2$

$\Rightarrow \quad D^2 = (x-4)^2 + 2x - 2x^2$

$= x^2 + 16 - 8x + 2x - 2x^2$

$= -x^2 - 6x + 16 \quad \ldots(i)$

On differentiating Eq. (i), we get

$2D\frac{dD}{dx} = -2x - 6 \quad \ldots(ii)$

$= -2(x+3)$

For maximum or minimum value,

put $\frac{dD}{dx} = 0$

$\therefore \quad x = -3$

Again, on differentiating Eq. (ii), we get

$\frac{d^2D}{dx^2}$ = negative on putting $x = -3$

$\therefore$ The longest distance is

$D^2 = -9 - 6(-3) + 16$

$= -9 + 18 + 16 = 25$

$\therefore \quad D = 5$ units

51 The given curve is

$5x^5 - 10x^3 + x + 2y + 4 = 0 \quad \ldots(i)$

On differentiating Eq. (i), we get

$25x^4 - 30x^2 + 1 + 2\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-25x^4 + 30x^2 - 1}{2}$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2}$ at P

$\therefore$ Slope of normal is 2.

Therefore, its equation is

$(y+2) = 2(x-0)$

$\Rightarrow \quad y = 2x - 2 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$5x^5 - 10x^3 + x + 4x - 4 + 4 = 0$

$\Rightarrow \quad 5x[x^4 - 2x^2 + 1] = 0$

$\Rightarrow \quad 5x(x^2 - 1)^2 = 0$

$\Rightarrow \quad x = 0$

or $\quad x^2 = 1$ or $x = 0, 1, -1$

$\therefore \quad y = -2, 0, -4$

Since, the other two points are $(1, 0), (-1, -4)$.

The tangents at these points are

$(y - 0) = 2(x - 1)$

and $(y + 4) = 2(x + 1)$ or $y = 2(x - 1)$

52 Let $I = \int_{-1}^{2} \{|x-1| + [x]\}\,dx$

$= \int_{-1}^{2} (|x-1|)\,dx + \int_{-1}^{2} [x]\,dx$

$= \int_{-1}^{1} -(x-1)\,dx + \int_{1}^{2} (x-1)\,dx$

$+ \int_{-1}^{2} [x]\,dx$

$I = I_1 + I_2$

where,

$I_1 = -\frac{1}{2}[(x-1)^2]_{-1}^{1} + \frac{1}{2}[(x-1)^2]_{1}^{2}$

$= \frac{1}{2}\{[(x-1)^2]_1^2 - [(x-1)^2]_{-1}^{1}\}$

$= \frac{1}{2}\{1 + 4\} = \frac{5}{2} \quad \ldots(i)$

and $I_2 = \int_{-1}^{0} -dx + \int_{0}^{1} 0 \cdot dx + \int_{1}^{2} dx$

$= -1 + 0 + 1 = 0 \quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$I = I_1 + I_2 = \frac{5}{2} + 0 = \frac{5}{2}$

53 According to the question,

$\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$

$\Rightarrow \quad \frac{xdy - ydx}{x} = -\left[\cos^2 \frac{y}{x}\right]dx$

$\Rightarrow \quad \sec^2 \frac{y}{x} \frac{(xdx - ydx)}{x^2} = -\frac{dx}{x}$

$\sec^2 \frac{y}{x} d\left(\frac{y}{x}\right) = -\frac{dx}{x} \quad \ldots(i)$

On integrating both sides of Eq. (i), we get

$\tan\left(\frac{y}{x}\right) = -\log x + C$

When $x = 1/2$ and $y = \pi/8$, then

$1 = -\log\frac{1}{2} + C = -[-\log 2] + C$

$1 - \log 2 = C$

$\therefore \tan\frac{y}{x} = -\log x + 1 - \log 2$

$= -\log 2x + \log e = \log\left(\frac{e}{2x}\right)$

$\Rightarrow \quad \frac{y}{x} = \tan^{-1}\left(\log\frac{e}{2x}\right)$

$\Rightarrow \quad y = x\tan^{-1}\left(\log\frac{e}{2x}\right)$

54. Given, $\frac{xdy - y\,dx}{y^2} = dy$

$\Rightarrow \quad d\left(\frac{x}{y}\right) = -dy$

$\Rightarrow \quad \frac{x}{y} = -y + C \quad$ [integrating]

As $y(1) = 1 \Rightarrow C = 2$

$\therefore \quad \frac{x}{y} + y = 2$

Again, for $x = -3$,

$-3 + y^2 = 2y$

$\Rightarrow (y+1)(y-3) = 0$

Also, $\quad y > 0$

$\Rightarrow \quad y = 3 \quad$ [neglecting $y = -1$]

55 Let $I = \int_{0}^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$

Put $\sin x - \cos x = t$

Then, $(\sin x + \cos x)\,dx = dt$

$\therefore I = \int_{-1}^{0} \frac{dt}{9 + 16(1 - t^2)}$

$= \int_{-1}^{0} \frac{dt}{25 - 16t^2}$

$= \frac{1}{10}\int_{-1}^{0}\left(\frac{1}{5-4t} + \frac{1}{5+4t}\right)dt$

$= \left[\frac{1}{10}\cdot\frac{1}{4}[\log(5+4t) - \log(5-4t)]\right]_{-1}^{0}$

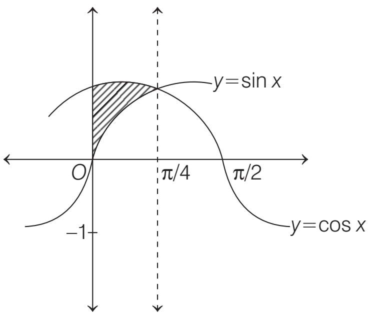
$= \frac{1}{40}(\log 9 - \log 1) = \frac{1}{20}\log 3$

56. $\int_{-2}^{3} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$

Since, $e^{\cos x}\sin x$ is an odd function.

$\therefore \quad \int_{-2}^{3} f(x)\,dx = 0 + 2(3 - 2) = 2$

57. Required area, $A = \int_{x_1}^{x_2} y\,dx$



$= \int_{0}^{\pi/4} \cos x\,dx - \int_{0}^{\pi/4} \sin x\,dx$

$= [\sin x]_0^{\pi/4} - [-\cos x]_0^{\pi/4}$

$= \left(\frac{1}{\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} - 1\right) = \sqrt{2} - 1$

58 The equation of the given curve is

$y = 7x^3 + 11 \quad \ldots(i)$

$\Rightarrow \quad \frac{dy}{dx} = 7 \times 3x^2 = 21x^2$

[differentiating w.r.t. x]

$\therefore$ Slope of tangent at $x = 2$ is

$\left(\frac{dy}{dx}\right)_{x=2} = 21(2)^2 = 84$

Slope of tangent at $x = -2$ is

$\left(\frac{dy}{dx}\right)_{x=-2} = 21(-2)^2 = 84$

It is observed that the slopes of the tangents at the points where, $x = 2$ and $x = -2$ are equal. Hence, the two tangents are parallel.

Hence, both the statements are true and Statement II is correct explanation of Statement I.

59 $f'(x) = \cos(\sqrt{x})^2 \dfrac{d}{dx}(\sqrt{x})$

$$- \cos\left(\frac{1}{x}\right)^2 \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{2}\frac{\cos x}{\sqrt{x}} + \cos\left(\frac{1}{x^2}\right)\frac{1}{x^2}$$

$$\Rightarrow \quad f'(1) = \frac{1}{2}\cos 1 + \cos 1$$

$$= \frac{3}{2}\cos 1$$

60 Given equation can be rewritten as

$$\frac{x}{y^{4/3}}\frac{dy}{dx} + 6y^{-1/3} = 3x$$

Put $\quad y^{-1/3} = v$

$$\Rightarrow \quad y^{-4/3}\frac{dy}{dx} = -3\frac{dv}{dx}$$

$$\therefore \quad \frac{dv}{dx} - \frac{2}{x}v = -1$$

Here, $\text{IF} = e^{\int -\frac{2}{x}dx} = x^{-2}$

$\therefore$ Solution is $x^{-2}v = \dfrac{1}{x} + C$

$$\Rightarrow \quad v = x + Cx^2$$

$$\Rightarrow \quad y(x) = \frac{1}{(x + Cx^2)^3}$$

61 $\displaystyle\int \frac{x^2 - 1}{(x^2 + 1)\sqrt{x^4 + 1}}dx$

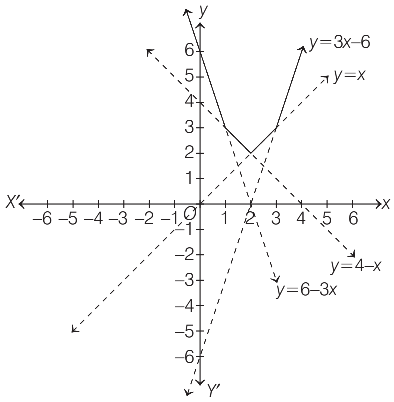
$$= \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)\sqrt{x^2 + \frac{1}{x^2}}}dx$$

$$= \int \frac{dt}{t\sqrt{t^2 - 2}} \quad \left[\text{put } t = x + \frac{1}{x}\right]$$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\frac{x^2 + 1}{x\sqrt{2}} + C$$

62 Let $f(x) = |x - 1| + |x - 2| + |x - 3|$

$$= \begin{cases} 6 - 3x, & x \le 1 \\ 4 - x, & 1 < x \le 2 \\ x, & 2 < x \le 3 \\ 3x - 6, & x > 3 \end{cases}$$



Clearly, the function has absolute minimum at $x = 2$.

So, the absolute minimum is equal to 2.

Also, the curve is taking sharp turn at $x = 1$, 2 and 3.

$\therefore f$ is not differentiable at $x = 1$, 2 and 3.

63 Given,

$f(x) = \max\{(x - 1)^2 + 1, |x - 1|\}$

$= (x - 1)^2 + 1$

$\therefore \quad f'(x) = 2(x - 1) = 0 \quad$ [say]

$\Rightarrow \quad x = 1 \in [0, 3]$

Now, $f(0) = 2$, $f(1) = 1$, $f(3) = 5$

$\therefore$ Greatest value of

$f(x) = \max\{f(0), f(1), f(3)\} = 5$

64 For vertical tangent, $\dfrac{dx}{d\theta} = 0$

$\therefore \; -3\cos\theta = 0 \Rightarrow \cos\theta = 0$

$$\Rightarrow \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

At $\theta = \dfrac{\pi}{2}$, $x = 2 - 3$

$= -1$, $y = 3 + 0 = 3$

i.e. $(-1, 3)$ and

At $\theta = \dfrac{3\pi}{2}$, $x = 2 + 3 = 5$

and $y = 3 + 0 = 3$

i.e. $(5, 3)$.

65 Suppose $f(x) = 0$ has a real root say $x = a$, then $f(x) < 0$ for all $x < a$. Thus, $|f(x)|$ becomes strictly decreasing on $(-\infty, a)$.

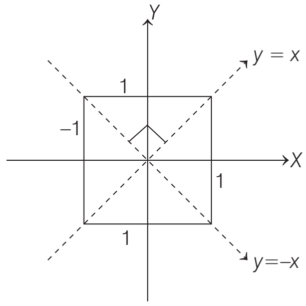
So, Statement I is true.

66 As the area enclosed by $|x| + |y| \le a$ is the area of square i.e. $2a^2$.

$\therefore$ Area enclosed by

$|x + y| + |x - y| \le 2$

is area of square shown as



$\therefore$ Required area

$$= 4\left(\frac{1}{2} \times 2 \times 1\right) = 4 \text{ sq units}$$

Also, the area enclosed by

$|x + y| + |x - y| \le 2$

is symmetric about X-axis, Y-axis, $y = x$ and $y = -x$.

Hence, both the statements are true but Statement II is not the correct explanation of Statement I.